Ejercicios Transformadas de Laplace:

Hallar la transformada de Laplace de la función f en cada uno de los siguientes casos:

1.- $f(t) = 4\sin t - \frac{1}{2}\cos t + e^{3t}$

$f(t)$, Laplace transform is: $\frac{4}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{s-3} = \frac{11s+s^2-22}{2(s-3)\left(s^2+1\right)}$

2.- $f(t) = \sin \alpha t$, Laplace transform is: $\frac{\alpha}{s^2+\alpha^2}$

3.- $f(t) = \cos \alpha t$, Laplace transform is: $\frac{s}{s^2+\alpha^2}$

4.- $f(t) = \sinh(\alpha t)$, Laplace transform is: $\frac{\alpha}{s^2-\alpha^2}$

5.- $f(t) = \cosh(\alpha t)$, Laplace transform is: $\frac{s}{s^2-\alpha^2}$

6.- $f(t) = e^{2t}(\cos t + \sin t) \rightarrow f(t) = e^{2t}\cos t + e^{2t}\sin t$,

Laplace transform is: $\frac{1}{(s-2)^2+1} + \frac{s-2}{(s-2)^2+1} = \frac{s-1}{s^2-4s+5}$

7.- $f(t) = t^2\sin 4t + te^{-t}$, Laplace transform is: $\frac{1}{(s+1)^2} + \frac{24s^2-128}{\left(s^2+16\right)^3}$

8.- $f(t) = t^8 - t^5 + 3t^2$, Laplace transform is: $\frac{6}{s^3} - \frac{120}{s^6} + \frac{40\,320}{s^9}$

9.- $f(t) = e^{\frac{5t}{2}}\cosh(\frac{7t}{2})$, Laplace transform is: $\frac{s-\frac{5}{2}}{\left(s-\frac{5}{2}\right)^2-\frac{49}{4}} = \frac{(2s-5)}{2(s+1)(s-6)}$

10.- $f(t) = te^{4t}$, Laplace transform is: $\frac{1}{(s-4)^2}$

11.- $f(t) = \frac{d}{dt}(te^{4t}) = e^{4t} + 4te^{4t}$

$f(t) = e^{4t} + 4te^{4t}$, Laplace transform is: $\frac{1}{s-4} + \frac{4}{(s-4)^2} = (s-4)^{-2}s = \frac{s}{(s-4)^2}$

12.- $f(t) = \left(\frac{d}{dt}te^t\right)$, Laplace transform is: $\frac{1}{s-1} + \frac{1}{(s-1)^2}$

$\frac{d}{dt}te^t = e^t + te^t$; $f(t) = e^t + te^t$

Laplace transform is: $\frac{1}{s-1} + \frac{1}{(s-1)^2} = (s-1)^{-2}s = \frac{s}{(s-1)^2}$

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13.- $f(t) = \frac{d^2}{dt^2}(\cos t + te^t)$

$\frac{d^2}{dt^2}(\cos t + te^t) = 2e^t - \cos t + te^t$

$f(t) = 2e^t - \cos t + te^t$, Laplace transform is: $\frac{2}{s-1} + \frac{1}{(s-1)^2} - \frac{s}{s^2+1}$

14.- $f(t) = \int_0^t \cos u du$, Laplace transform is: $\frac{1}{s^2+1}$

en los ejercicios que siguen, el resultado se da a continuación del signo igual.

15.- $\mathcal{L}(t^{10}e^{2t}) = \frac{3628\,800}{(s-2)^{11}}$

16.- $\mathcal{L}\left(\frac{1}{2}\cosh 5t\right) = \frac{1}{2}\frac{s}{s^2-25}$

17.- $\mathcal{L}(e^{2t}(t-1)^3) = \mathcal{L}\left[3te^{2t} - e^{2t} - 3t^2e^{2t} + t^3e^{2t}\right] =$

$\frac{3}{(s-2)^2} - \frac{1}{s-2} - \frac{6}{(s-2)^3} + \frac{6}{(s-2)^4}$

18.- $\mathcal{L}(e^{3t}\cos^2 2t) = \frac{1}{s-3}\frac{(s-3)^2+8}{(s-3)^2+16}$

19.- $\mathcal{L}\left(2e^{-3t}\cos 3t + t^4e^{\pi t} + \frac{1}{2}t\cosh^2 t\right) = \frac{1}{2}\mathcal{L}\left(t\cosh^2 t\right) + \frac{24}{(s-\pi)^5} + 2\frac{s+3}{(s+3)^2+9}$

19.1 $\mathcal{L}\left[t\cosh^2 t\right] = \mathcal{L}\left[t\left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right)^2\right] = \mathcal{L}\left[\frac{1}{2}t + \frac{1}{4}\frac{t}{e^{2t}} + \frac{1}{4}te^{2t}\right] =$

$\frac{1}{2s^2} + \frac{1}{4(s-2)^2} + \frac{1}{4}\mathcal{L}\left[\frac{t}{e^{2t}}\right]$

19.2 $\mathcal{L}\left[\frac{t}{e^{2t}}\right] = \frac{1}{(s+2)^2}$;

$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer

20.- $\mathcal{L}\left(\int_0^t \cos u du\right) = \frac{1}{s^2+1}$

21.- $\mathcal{L}\left(\int_0^t e^{au}du\right) = \frac{1}{s}\mathcal{L}[e^{at}] - \frac{1}{s}\int_0^0 f(u)du = \frac{1}{s(s-a)}$

haciendo uso de la propiedad: $\mathcal{L}\left[\int_a^t f(u)du\right] = \frac{1}{s}\mathcal{L}[f(t)] - \frac{1}{s}\int_0^a f(u)du$

22.- $\mathcal{L}\left[-te^{-5t}\cosh^3 t\right] = \mathcal{L}\left[-\frac{1}{8}\frac{t}{e^{2t}} - \frac{3}{8}\frac{t}{e^{4t}} - \frac{3}{8}\frac{t}{e^{6t}} - \frac{1}{8}\frac{t}{e^{8t}}\right] =$

$-\frac{1}{8}\mathcal{L}\left(\frac{t}{e^{2t}}\right) - \frac{3}{8}\mathcal{L}\left(\frac{t}{e^{4t}}\right) - \frac{3}{8}\mathcal{L}\left(\frac{t}{e^{6t}}\right) - \frac{1}{8}\mathcal{L}\left(\frac{t}{e^{8t}}\right)$

$-te^{-5t}\cosh^3 t = -\frac{t}{e^{5t}}\cosh^3 t = -\frac{t}{e^{5t}}\left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right)^3 =$
$-\frac{1}{8}\frac{t}{e^{2t}} - \frac{3}{8}\frac{t}{e^{4t}} - \frac{3}{8}\frac{t}{e^{6t}} - \frac{1}{8}\frac{t}{e^{8t}}$

por la propiedad anterior:
$-\frac{1}{8}\mathcal{L}\left(\frac{t}{e^{2t}}\right) - \frac{3}{8}\mathcal{L}\left(\frac{t}{e^{4t}}\right) - \frac{3}{8}\mathcal{L}\left(\frac{t}{e^{6t}}\right) - \frac{1}{8}\mathcal{L}\left(\frac{t}{e^{8t}}\right)$

Guía de ejercicios:(se sugiere que trabaje con el SWP y ocupando las definiciones y realizando los cálculos pertinentes)
Hallar la transformada de Laplace de cada una de la funciones siguientes:

1.- $f(t) = sin(t+a)$..........$\mathcal{L}(\sin(t+a)) = \frac{\cos a + s\sin a}{s^2+1}$
2.- $f(t) = (t+a)^2$........... $\mathcal{L}((t+a)^2) = \frac{2}{s^3} + 2\frac{a}{s^2} + \frac{a^2}{s}$
3.- $f(t) = \cosh at$....................$\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$

4.- $f(t) = t^2e^t$....................$\mathcal{L}(t^2e^t) = \frac{2}{(s-1)^3}$

5.- $f(t) = t\sin t$....................$\mathcal{L}(t\sin t) = 2\frac{s}{\left(s^2+1\right)^2}$

6.- $f(t) = t^2\sin t$....................$\mathcal{L}(t^2\sin t) = 8\frac{s^2}{\left(s^2+1\right)^3} - \frac{2}{\left(s^2+1\right)^2}$

Resolver las siguientes ecuaciones diferenciales usando la T.L.

1.- $y'' - 3y' + 2y = 0$ .... con $y(0) = 3$; $y'(0) = 4$...sol: $y = e^{2t} + 2e^t$
2.- $y'' + y = 0$ .... con $y(0) = -1$; $y'(0) = 3$...sol: $y(x) = 3\sin x$
3.- $y''' + y'' + 4y' + 4y = -2$ .... con $y(0) = 0$; $y'(0) = 1$;$y''(0) = -1$

Exact solution is: $y(x) = -\frac{1}{5}\cos 2x + \frac{1}{5}e^{-x} + \frac{1}{10}\sin 2x - \frac{1}{2}$

4.- Resuelva algunas de las E.D. de la guía de Ecuaciones diferenciales y compruebe

Encontrar la inversa de la T.L. de cada una de las funciones siguientes:

1.- $\frac{1}{s(s+1)}$ ....$\mathcal{L}^{-1}(\frac{1}{s(s+1)}) = 1 - e^{-t}$

2.- $\frac{3}{(s-1)^2}$..$\mathcal{L}^{-1}(\frac{3}{(s-1)^2}) = 3te^{t}$

3.- $\frac{1}{s(s+2)^2}$.........................$\mathcal{L}^{-1}(\frac{1}{s(s+2)^2}) = \frac{1}{4} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$

4.- $\frac{5}{s^2(s-5)^2}$.......................$\mathcal{L}^{-1}(\frac{5}{s^2(s-5)^2}) = \frac{1}{5}t - \frac{2}{25}e^{5t} + \frac{1}{5}te^{5t} + \frac{2}{25}$

5.- $\frac{1}{s^2+4s+29}$......................$\mathcal{L}^{-1}(\frac{1}{s^2+4s+29}) = \frac{1}{5}(\sin 5t)e^{-2t}$

6.- $\frac{1}{s^2+3s}$...$\mathcal{L}^{-1}(\frac{1}{s^2+3s}) = \frac{1}{3} - \frac{1}{3}e^{-3t}$

7.- $\frac{s}{s^2+1}$...$\mathcal{L}^{-1}(\frac{s}{s^2+1}) = \cos t$

8.- $\frac{s-1}{s^2+1}$...$\mathcal{L}^{-1}(\frac{s-1}{s^2+1}) = \cos t - \sin t$

9.- $\frac{s}{s^3+1}$...........$\mathcal{L}^{-1}(\frac{s}{s^3+1}) = \frac{1}{3}e^{\frac{1}{2}t}\left(\cos\frac{1}{2}t\sqrt{3} + \sqrt{3}\sin\frac{1}{2}t\sqrt{3}\right) - \frac{1}{3}e^{-t}$

10.- $\frac{5}{s^2(s-5)}$......................$\mathcal{L}^{-1}(\frac{5}{s^2(s-5)}) = \frac{1}{5}e^{5t} - t - \frac{1}{5}$

Exploración de nuevas aplicaciones e invención de ejercicios para la Transformada de Laplace

En los ejemplos siguientes se puede observar la posibilidad de calcular integrales usando la T.L.

1.- $\mathcal{L}\left(\int_0^t u\,du\right) = \frac{1}{s}\mathcal{L}(t) = \frac{1}{s^3}$ ...... $\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2}t^2$

2.- $\mathcal{L}\left(\int_0^t u^2du\right) = \frac{1}{s}\mathcal{L}\left(t^2\right) = \frac{2}{s^4}$........... $\mathcal{L}^{-1}\left[\frac{2}{s^4}\right] = \frac{1}{3}t^3$

3.- Calcular $\int_0^t ue^{u}du$ aplicando la T.L.

$$\mathcal{L}\left(\int_0^t ue^u du\right) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}\right) = te^t - e^t + 1$$

comprobación: $\frac{d}{dt}(te^t - e^t + 1) = te^t$

4.- Calcular $\int_0^t e^u \sin u du$ aplicando la T.L.

$$\mathcal{L}\left(\int_0^t e^u \sin u du\right) = \frac{1}{s}\mathcal{L}(e^t \sin t) = \frac{1}{s\left((s-1)^2+1\right)} = \frac{1}{2s} + \frac{1}{2}\frac{2-s}{s^2-2s+2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s} + \frac{1}{2}\frac{2-s}{s^2-2s+2}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2-s}{s^2-2s+2}\right)$$

$$\int_0^t e^u \sin u du = \frac{1}{2} + \frac{1}{2}e^t(\sin t - \cos t)$$

comprobación: $\frac{d}{dt}(\frac{1}{2} + \frac{1}{2}e^t(\sin t - \cos t)) = (\sin t)e^t$

5.- Usando la regla más general: $\mathcal{L}\left[\int_a^t f(u)du\right] = \frac{1}{s}\mathcal{L}[f(t)] - \frac{1}{s}\int_0^a f(u)du$

Calcular: $\int_a^t \cos 3u du$

$$\mathcal{L}\left[\int_a^t \cos 3u du\right] = \frac{1}{s}\mathcal{L}[\cos 3t] - \frac{1}{s}\int_0^a \cos 3u du = \frac{1}{s^2+9} - \frac{1}{3s}\sin 3a$$

$\frac{1}{s}\mathcal{L}[\cos 3t] = \frac{1}{s^2+9}$ $\qquad$ $\frac{1}{s}\int_0^a \cos 3u du = \frac{1}{3s}\sin 3a$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9} - \frac{1}{3s}\sin 3a\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) - \frac{\sin 3a}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$\int_a^t \cos 3u du = \frac{1}{3}\sin 3t - \frac{1}{3}\sin 3a$$

$$\mathcal{L}\left(\int_a^t e^u \sin u du\right) = \frac{1}{2s}(\cos a)e^a - \frac{1}{2s}(\sin a)e^a + \frac{1}{2\left((s-1)^2+1\right)} - \frac{1}{2}\frac{s-1}{(s-1)^2+1}$$

1.- $\mathcal{L}[e^{-t}(3\sinh 2t - 5\cosh 2t)]$

$3e^{-t}\sinh 2t - 5e^{-t}\cosh 2t = 3e^{-t}\left(\frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}\right) - 5e^{-t}\left(\frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t}\right)$al

simplificar obtenemos:

$$3e^{-t}\sinh 2t - 5e^{-t}\cosh 2t = -(e^{-t})\left(4e^{-2t} + e^{2t}\right) = -e^{t} - \frac{4}{e^{3t}}$$

$$\mathcal{L}[e^{-t}(3\sinh 2t - 5\cosh 2t)] = \mathcal{L}\left[-e^{t} - \frac{4}{e^{3t}}\right] = -\mathcal{L}[e^{t}] - 4\mathcal{L}\left[e^{-3t}\right]$$

$\mathcal{L}[e^{-t}(3\sinh 2t - 5\cosh 2t)]$

$$\mathcal{L}[e^{-t}(3\sinh 2t - 5\cosh 2t)] = -\frac{1}{s-1} - \frac{4}{s+3} = \frac{(1-5s)}{(s-1)(s+3)}$$

otra forma:

como : $\mathcal{L}[\sinh 2t] = \frac{2}{s^2-4}$; $\mathcal{L}[\cosh 2t] : \frac{s}{s^2-4}$

y además: $\mathcal{L}[f(t)] = F(s) \rightarrow \mathcal{L}[e^{at}f(t)] = F(s-a)$

$$\mathcal{L}[e^{-t}(3\sinh 2t - 5\cosh 2t)] = 3\mathcal{L}[e^{-t}\sinh 2t)] - 5\mathcal{L}[e^{-t}\cosh 2t]$$

$$3\mathcal{L}[e^{-t}\sinh 2t)] = \frac{6}{(s+1)^2-4}$$

$$5\mathcal{L}[e^{-t}\cosh 2t] = 5\frac{s+1}{(s+1)^2-4}$$

$$\frac{6}{(s+1)^2-4} - 5\frac{s+1}{(s+1)^2-4} = \frac{6-5(s+1)}{(s+1)^2-4} = \frac{1-5s}{2s+s^2-3}$$

Notabene:

$(s+1)^2 - 4 = 2s + s^2 - 3 = (s+3)(s-1)$

2.- Si $\mathcal{L}[f(t)] = \frac{e^{\frac{-1}{s}}}{s}$; hallar $\mathcal{L}[e^{-t}f(3t)]$

Se tiene que $\mathcal{L}[f(3t)] = \frac{1}{3}\frac{e^{\frac{-1}{\left(\frac{s}{3}\right)}}}{\left(\frac{s}{3}\right)} = \frac{e^{\frac{-3}{s}}}{s}$

$\mathcal{L}[f(3t)] = \frac{e^{\frac{-3}{s}}}{s} \rightarrow \mathcal{L}[e^{-t}f(3t)] = \frac{\mathrm{e}^{\frac{-3}{s+1}}}{s+1}$

otra forma: $\mathcal{L}[e^{-t}f(t)] = \frac{e^{\frac{-1}{s+1}}}{s+1} \rightarrow \mathcal{L}[e^{-t}f(3t)] = \frac{e^{\frac{-1}{\frac{s}{3}+1}}}{\frac{s}{3}+1}$

$\mathcal{L}[e^{-t}f(3t)] = \frac{1}{3}\frac{e^{\frac{-3}{s+3}}}{\frac{s}{3}+1} = \frac{e^{\frac{-3}{s+3}}}{s+3}$

2.- Si $F(s) = \mathcal{L}[f(t)]$; demuestre que para $r > 0$ se cumple:

$\mathcal{L}[r^t f(at)] = \frac{1}{s-\ln r}F(\frac{s-\ln r}{a})$

sea $y = r^t \rightarrow \ln y = t\ln r \rightarrow y = e^{t\ln r} \rightarrow r^t = e^{t\ln r}$

por la propiedad de traslación:

$\mathcal{L}[r^t f(t)] = \mathcal{L}\left[e^{t\ln r}f(t)\right] = F(s-\ln r)$

Aceptando que : $\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{s}{a})$ (propiedad de cambio de escala)

$\mathcal{L}[r^t f(at)] = \mathcal{L}\left[e^{t\ln r}f(at)\right] = \frac{1}{a}F(\frac{s-\ln r}{a})$

veamos un ejemplo de cómo funciona esta propiedad:

definimos las siguientes funciones : $f(t) = \sin t \quad F(s) = \frac{1}{s^2+1}$

$\mathcal{L}[f(at)] = \frac{a}{a^2+s^2}$

$F(\frac{s-\ln r}{a}) = \frac{a^2}{(s-\ln r)^2+a^2}$

$\frac{1}{a}F(\frac{s-\ln r}{a}) = \frac{a}{(s-\ln r)^2+a^2}$

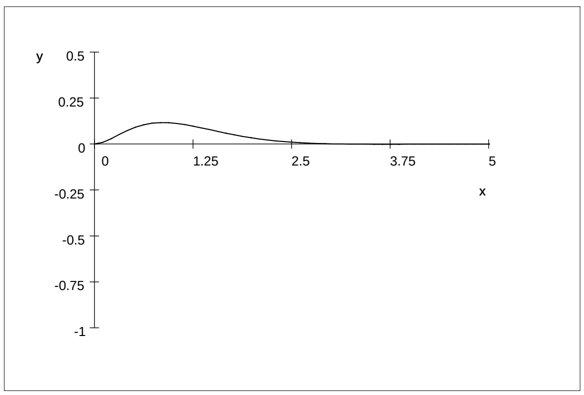
$\mathcal{L}\left[e^{t\ln r}f(at)\right] = \dfrac{a}{a^2 + (s - \ln r)^2}$

$\mathcal{L}\left[e^{t\ln r}f(at)\right] = \frac{1}{a}F(\dfrac{s - \ln r}{a})$ is true

$f(t) = \sin t$

$\int_0^\infty e^{-2t}f(t)dt = \dfrac{1}{5} = \dfrac{1}{s^2+1} + \lim_{t\to\infty} \dfrac{1}{e^{st} + s^2 e^{st}}(-\cos t - s\sin t)$

$e^{-2t}f(t)$



Pero también podemos usar la T.L. para calcular derivadas, veamos cómo:

1.- Calcular $\frac{d}{dt}(f(t))$ siendo $f(t) = t^4$

$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\mathcal{L}[f(t)] - f(o) \to \mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\frac{24}{s^5} - 0 \to \frac{d}{dt}f(t) = \mathcal{L}^{-1}\left[\frac{24}{s^4}\right] = 4t^3$

$\mathcal{L}[f(t)] = \frac{24}{s^5}$ $f(0) = 0$ ¿procedimiento muy largo? Pero sirve para ilustrar esta propiedad.

2.- Calcular $\frac{d}{dt}(f(t))$ siendo $f(t) = t\sin t$

$\mathcal{L}\left[\frac{d}{dt}(f(t))\right] = s\mathcal{L}[f(t)] - f(0) \to \mathcal{L}\left[\frac{d}{dt}(f(t))\right] = s\left(2\frac{s}{\left(s^2+1\right)^2}\right) - 0$

$$\mathcal{L}[f(t)] = 2\frac{s}{\left(s^2+1\right)^2}$$

$$\frac{d}{dt}(f(t)) = \mathcal{L}^{-1}\left[\frac{2s^2}{\left(s^2+1\right)^2}\right] = \mathcal{L}^{-1}\left[\frac{2}{s^2+1} - \frac{2}{\left(s^2+1\right)^2}\right]$$

$$\frac{d}{dt}(f(t)) = 2\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{\left(s^2+1\right)^2}\right] = 2\sin t - (\sin t - t\cos t)$$

finalmente : $\frac{d}{dt}(f(t)) = \sin t + t\cos t$

Notabene: $\frac{2s^2}{\left(s^2+1\right)^2} = \frac{2}{s^2+1} - \frac{2}{\left(s^2+1\right)^2}$(el proceso es un poquito largo y complicado pero vale la pena realizarlo para familiarizarnos con esta propiedad.)

Ejercicios de T.L.

1.- $y'' - 3y' + 2y = 0$; $y(0) = 0$; $y'(0) = 1$

$f^{(n)}(t) \quad s^n F(s) - s^{(n-1)} f(0) - \ldots - f^{(n-1)}(0)$

$$\mathcal{L}\left[y'' - 3y' + 2y\right] = \mathcal{L}[0]$$

$$\mathcal{L}\left[y''\right] - 3\mathcal{L}\left[y'\right] + 2\mathcal{L}[y] = 0$$

$$s^2\mathcal{L}[y] - sy(0) - y'(0) - 3(s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] = 0$$

$$s^2\mathcal{L}[y] - 0 - 1 - 3(s\mathcal{L}[y] - 0) + 2\mathcal{L}[y] = 0$$

$$s^2\mathcal{L}[y] - 1 - 3s\mathcal{L}[y] + 2\mathcal{L}[y] = 0$$

$$\mathcal{L}[y]\left(s^2 - 3s + 2\right) = 1$$

$$\mathcal{L}[y] = \frac{1}{\left(s^2 - 3s + 2\right)}$$

$$\frac{1}{\left(s^2 - 3s + 2\right)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{s-1}$$

$$y = \mathcal{L}^{-1}\left[\frac{1}{s-2} - \frac{1}{s-1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$\mathcal{L}^{-1}\left[\frac{1}{s-2} \right] = e^{2t}$

$\mathcal{L}^{-1}\left[\frac{1}{s-1} \right] = e^{t}$

$y(t) = e^{2t} - e^{t}$

Comprobación: $\frac{dy(t)}{dt} = 2e^{2t} - e^{t}$

$\frac{d^2y(t)}{dt^2} = 4e^{2t} - e^{t}$

$y'' - 3y' + 2y = 0 \rightarrow$

$4e^{2t} - e^{t} - 3\left(2e^{2t} - e^{t}\right) + 2\left(e^{2t} - e^{t}\right) = 0$; se comprueba la solución...

2.- Demuestre que : $\mathcal{L}^{-1}\left[\frac{s^2+1}{s^3+4s} \right] = \frac{3}{4}\cos 2t + \frac{1}{4}$

descomponiendo en una suma de fracciones: $\frac{s^2+1}{s^3+4s} = \frac{1}{4s} + \frac{3}{4}\frac{s}{s^2+4}$

$\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s} \right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+4} \right] = \frac{1}{4} + \frac{3}{4}\cos 2t$

observación: $\mathcal{L}^{-1}\left[\frac{s}{s^2+4} \right] = \cos 2t$

3.- Calcular: $\mathcal{L}^{-1}\left[\frac{1}{s}\frac{s-a}{s+a} \right]$

$\mathcal{L}^{-1}\left[\frac{1}{s}\frac{s-a}{s+a} \right] = \mathcal{L}^{-1}\left[\frac{2}{a+s} - \frac{1}{s} \right] = \mathcal{L}^{-1}\left[\frac{2}{a+s} \right] - \mathcal{L}^{-1}\left[\frac{1}{s} \right] = 2e^{-at} - 1$

observación: $\frac{1}{s}\frac{s-a}{s+a} = \frac{2}{a+s} - \frac{1}{s}$

$\mathcal{L}^{-1}\left[\frac{2}{a+s} \right] = 2e^{-at}$; $\mathcal{L}^{-1}\left[\frac{1}{s} \right] = 1$

finalmente: $\mathcal{L}^{-1}\left[\frac{1}{s}\frac{s-a}{s+a} \right] = 2e^{-at} - 1$

otra forma: sea $\mathcal{L}[f(t)] = \frac{1}{s}\frac{s-a}{s+a} = \frac{1}{s}(1 - \frac{2a}{s+a}) = \frac{1}{s} - 2\frac{a}{s(a+s)}$

como: $\mathcal{L}[e^{-at}] = \frac{1}{a+s}$; $\mathcal{L}[1] = \frac{1}{s}$

$$\mathcal{L}[f(t)] = \frac{1}{s} - 2\frac{a}{s(a+s)} = \mathcal{L}[1] - 2a\frac{1}{s}\mathcal{L}[e^{-at}]$$

Transformada de una integral:

Sea $g(t) = \int_0^t f(u)du$ se cumple : $\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(u)du\right) = \frac{1}{s}\mathcal{L}(f(t))$

$$\mathcal{L}[f(t)] = \mathcal{L}[1] - 2a\mathcal{L}\left[\int_0^t e^{-au}du\right]$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[1 - 2a\int_0^t e^{-au}du\right] \to f(t) = 1 - 2a\int_0^t e^{-au}du$$

en donde : $\int e^{-au}du = -\frac{1}{a}e^{-au}$

$$f(t) = 1 - 2a\left[-\frac{1}{a}e^{-au}\right]_0^t = 1 + 2a\left(\frac{1}{a}e^{-at} - \frac{1}{a}e^0\right) = \frac{2}{e^{at}} - 1$$

finalmente: $f(t) = \frac{2}{e^{at}} - 1$

comprobación: $f(t) = 2e^{-at} - 1$; $\mathcal{L}[f(t)] = \frac{2}{a+s} - \frac{1}{s} = \frac{(s-a)}{s(a+s)}$

Sistemas de ecuaciones:

1.-
$$x(0) = 0$$
$$y(0) = 0$$
$$x' - 4x - y + 36t = 0$$
$$y' + 2x - y + 2e^t = 0$$

$$\mathcal{L}[x' - 4x - y + 36t] = \mathcal{L}[0] \to \mathcal{L}[x'] - 4\mathcal{L}[x] - \mathcal{L}[y] + 36\mathcal{L}[t] = 0 \to$$

$$s\mathcal{L}[y] - y(0) - 4\mathcal{L}[x] - \mathcal{L}[y] + 36\frac{1}{s^2} = 0 \to s\mathcal{L}[y] - 4\mathcal{L}[x] - \mathcal{L}[y] = -\frac{36}{s^2}$$

$$\mathcal{L}[y' + 2x - y + 2e^t] = \mathcal{L}[0] \to \mathcal{L}[y'] + 2\mathcal{L}[x] - \mathcal{L}[y] + 2\mathcal{L}[e^t] = 0 \to$$

$$s\mathcal{L}[y] - y(0) + 2\mathcal{L}[x] - \mathcal{L}[y] + 2\,\frac{1}{s-1} = 0 \to s\mathcal{L}[y] + 2\mathcal{L}[x] - \mathcal{L}[y] = -\frac{2}{s-1}$$

$$(s-1)\mathcal{L}[y] - 4\mathcal{L}[x] = \frac{-36}{s^2}$$

$$(s-1)\mathcal{L}[y] + 2\mathcal{L}[x] = -\frac{2}{s-1}$$

$$\begin{matrix}(s-1)y - 4x = \dfrac{-36}{s^2} \\ (s-1)y + 2x = -\dfrac{2}{s-1}\end{matrix}$$, Solution is: $\left[x = \dfrac{18s - s^2 - 18}{3s^3 - 3s^2}, y = \dfrac{36 - 4s^2 - 36s}{3s^2 - 6s^3 + 3s^4}\right]$

$$\mathcal{L}[x] = \frac{18s - s^2 - 18}{3s^3 - 3s^2}$$

$$\mathcal{L}[y] = \frac{36 - 4s^2 - 36s}{3s^2 - 6s^3 + 3s^4}$$

$$\mathcal{L}[x] = \frac{18s - s^2 - 18}{3s^3 - 3s^2} \to x(t) = \mathcal{L}^{-1}\left[\frac{18s - s^2 - 18}{3s^3 - 3s^2}\right] = 6t - \frac{1}{3}e^t$$

$$\mathcal{L}[y] = \frac{36 - 4s^2 - 36s}{3s^2 - 6s^3 + 3s^4} \to y(t) = \mathcal{L}^{-1}\left[\frac{36 - 4s^2 - 36s}{3s^2 - 6s^3 + 3s^4}\right] =$$

$$12t - 12e^t - \frac{4}{3}te^t + 12$$

luego:

$$x(t) = 6t - \frac{1}{3}e^t$$

$$y(t) = 12t - 12e^t - \frac{4}{3}te^t + 12$$

comprobación:

$$x' - 4x - y + 36t = 0$$

$$y' + 2x - y + 2e^t = 0$$

$$x'(t) - 4x(t) - y(t) + 36t = 13e^t + \frac{4}{3}te^t - 6$$

$$y'(t) + 2x(t) - y(t) + 2e^t = 0$$

Ejercicios Transformadas de Laplace:

Hallar la transformada de Laplace de la función f en cada uno de los siguientes casos:

1.- $f(t) = 4\sin t - \frac{1}{2}\cos t + e^{3t}$

$f(t)$, Laplace transform is: $\frac{4}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{s-3} = \frac{11s+s^2-22}{2(s-3)\left(s^2+1\right)}$

2.- $f(t) = \sin \alpha t$, Laplace transform is: $\frac{\alpha}{s^2+\alpha^2}$

3.- $f(t) = \cos \alpha t$, Laplace transform is: $\frac{s}{s^2+\alpha^2}$

4.- $f(t) = \sinh(\alpha t)$, Laplace transform is: $\frac{\alpha}{s^2-\alpha^2}$

5.- $f(t) = \cosh(\alpha t)$, Laplace transform is: $\frac{s}{s^2-\alpha^2}$

6.- $f(t) = e^{2t}(\cos t + \sin t) \to f(t) = e^{2t}\cos t + e^{2t}\sin t$,

Laplace transform is: $\frac{1}{(s-2)^2+1} + \frac{s-2}{(s-2)^2+1} = \frac{s-1}{s^2-4s+5}$

7.- $f(t) = t^2 \sin 4t + te^{-t}$, Laplace transform is: $\frac{1}{(s+1)^2} + \frac{24s^2-128}{\left(s^2+16\right)^3}$

8.- $f(t) = t^8 - t^5 + 3t^2$, Laplace transform is: $\frac{6}{s^3} - \frac{120}{s^6} + \frac{40\,320}{s^9}$

9.- $f(t) = e^{\frac{5t}{2}}\cosh(\frac{7t}{2})$, Laplace transform is: $\frac{s-\frac{5}{2}}{\left(s-\frac{5}{2}\right)^2 - \frac{49}{4}} = \frac{(2s-5)}{2(s+1)(s-6)}$

10.- $f(t) = te^{4t}$, Laplace transform is: $\frac{1}{(s-4)^2}$

11.- $f(t) = \frac{d}{dt}(te^{4t}) = e^{4t} + 4te^{4t}$

$f(t) = e^{4t} + 4te^{4t}$, Laplace transform is: $\frac{1}{s-4} + \frac{4}{(s-4)^2} = (s-4)^{-2}s = \frac{s}{(s-4)^2}$

12.- $f(t) = \left(\frac{d}{dt}te^t\right)$, Laplace transform is: $\frac{1}{s-1} + \frac{1}{(s-1)^2}$

$\frac{d}{dt}te^t = e^t + te^t$; $f(t) = e^t + te^t$

Laplace transform is: $\frac{1}{s-1} + \frac{1}{(s-1)^2} = (s-1)^{-2}s = \frac{s}{(s-1)^2}$

13.- $f(t) = \frac{d^2}{dt^2}(\cos t + te^t)$

$\frac{d^2}{dt^2}(\cos t + te^t) = 2e^t - \cos t + te^t$

$f(t) = 2e^t - \cos t + te^t$, Laplace transform is: $\frac{2}{s-1} + \frac{1}{(s-1)^2} - \frac{s}{s^2+1}$

14.- $f(t) = \int_0^t \cos u du$, Laplace transform is: $\frac{1}{s^2+1}$

15.- $\mathcal{L}(t^{10}e^{2t}) = \frac{3628\,800}{(s-2)^{11}}$

16.- $\mathcal{L}\left(\frac{1}{2}\cosh 5t\right)$

17.- $\mathcal{L}(e^{2t}(t-1)^3)$

18.- $\mathcal{L}(e^{3t}\cos^2 2t)$

19.- $\mathcal{L}\left(2e^{-3t}\cos 3t + t^4 e^{\pi t} + \frac{1}{2}t\cosh^2 t\right)$

20.- $\mathcal{L}\left(\int_0^t \cos u du\right)$

21.- $\mathcal{L}\left(\int_0^t e^{au} du\right)$

22.- $\mathcal{L}\left(-te^{-5t}\cosh^3 t\right)$

Guía de ejercicios:(se sugiere que trabaje con el SWP y ocupando las definicionesy realizando los cálculos pertinentes)

Hallar la transformada de Laplace de cada una de la funciones siguientes:

1.- $f(t) = sin(t+a)$...........$\mathcal{L}(\sin(t+a)) = \frac{\cos a + s\sin a}{s^2+1}$

2.- $f(t) = (t+a)^2$............ $\mathcal{L}((t+a)^2) = \frac{2}{s^3} + 2\frac{a}{s^2} + \frac{a^2}{s}$

$\mathcal{L}((t+a)^2) = \mathcal{L}(2at + a^2 + t^2) = 2a\mathcal{L}(t) + a^2\mathcal{L}(1) + \mathcal{L}(t^2) = 2\frac{a}{s^2} + \frac{a^2}{s} + \frac{2}{s^3}$

3.- $f(t) = \cosh at$.....................$\mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}$

4.- $f(t) = t^2 e^t$.....................$\mathcal{L}(t^2 e^t) = \frac{2}{(s-1)^3}$

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer

5.- $f(t) = t\sin t$.....................$\mathcal{L}(t\sin t) = 2\frac{s}{\left(s^2+1\right)^2}$

6.- $f(t) = t^2\sin t$.....................$\mathcal{L}(t^2\sin t) = 8\frac{s^2}{\left(s^2+1\right)^3} - \frac{2}{\left(s^2+1\right)^2}$

Resolver las siguientes ecuaciones diferenciales usando la T.L.

1.- $y'' - 3y' + 2y = 0$ .... con $y(0) = 3$; $y'(0) = 4$...sol: $y = e^{2t} + 2e^t$

2.- $y'' + y = 0$ .... con $y(0) = -1$; $y'(0) = 3$...sol:
$\mathcal{L}(y'' + y) = \mathcal{L}(0) \to \mathcal{L}(y'') + \mathcal{L}(y) = 0$

$s^2F(s) - sy(0) - y'(0) = 0$
$s^2F(s) - s(-1) - 3 + F(s) = 0 \to s^2F(s) + s - 3 + F(s) = 0 \to F(s) = \frac{3-s}{s^2+1}$
$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{3-s}{s^2+1}\right) = 3\sin t - \cos t$
$f^{(n)}(t) \quad s^nF(s) - s^{(n-1)}f(0) - \ldots - f^{(n-1)}(0)$

3.- $y''' + y'' + 4y' + 4y = -2$ .... con $y(0) = 0$; $y'(0) = 1$; $y''(0) = -1$
$\mathcal{L}\left(y''' + y'' + 4y' + 4y\right) = \mathcal{L}(-2)$
$\mathcal{L}(y''') + \mathcal{L}\left(y''\right) + 4\mathcal{L}\left(y'\right) + 4\mathcal{L}(y) = -2\mathcal{L}(1)$
$\mathcal{L}(y''') = s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0) = s^3\mathcal{L}(y) - 0 - s - (-1) =$

$s^3\mathcal{L}(y) - s + 1$

$\mathcal{L}\left(y''\right) = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - 0 - 1 = s^2\mathcal{L}(y) - 1$

$4\mathcal{L}\left(y'\right) = 4s\mathcal{L}(y) - 4y(0) = 4s\mathcal{L}(y) - 0 = 4s\mathcal{L}(y)$

$4\mathcal{L}(y) = 4\mathcal{L}(y)$

$\mathcal{L}(-2) = -\frac{2}{s}$

$s^3\mathcal{L}(y) - s + 1 + s^2\mathcal{L}(y) - 1 + 4s\mathcal{L}(y) + 4\mathcal{L}(y) = -\frac{2}{s}$

$s^3\mathcal{L}(y) + s^2\mathcal{L}(y) + 4s\mathcal{L}(y) + 4\mathcal{L}(y) = -\frac{2}{s} + s - 1 + 1$

$\left(s^3 + s^2 + 4s + 4\right)\mathcal{L}(y) = -\frac{2}{s} + s$

$$F(s) = \frac{-\frac{2}{s} + s}{\left(s^3 + s^2 + 4s + 4\right)} \to F(s) = \frac{s^2 - 2}{s\left(s^3 + s^2 + 4s + 4\right)}$$

$$\frac{s^2 - 2}{s\left(s^3 + s^2 + 4s + 4\right)} = \frac{1}{5(s+1)} - \frac{1}{2s} + \frac{6}{5\left(s^2 + 4\right)} + \frac{3}{10}\frac{s}{s^2 + 4}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{5(s+1)} - \frac{1}{2s} + \frac{6}{5\left(s^2+4\right)} + \frac{3}{10}\frac{s}{s^2+4}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{5(s+1)}\right) - \mathcal{L}^{-1}\left(\frac{1}{2s}\right) + \mathcal{L}^{-1}\left(\frac{6}{5\left(s^2+4\right)}\right) + \mathcal{L}^{-1}\left(\frac{3}{10}\frac{s}{s^2+4}\right)$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{(s+1)}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{6}{5}\mathcal{L}^{-1}\left(\frac{1}{\left(s^2+4\right)}\right) + \frac{3}{10}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{2} + \frac{3}{5}\sin 2t + \frac{3}{10}\cos 2t$$

$y''' + y'' + 4y' + 4y = -2$

$\frac{d^3}{dt^3}y(t) + \frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 4y(t) = -2$ (esto ha sido una comprobación)

....

4.- Resuelva algunas de las E.D. de la guía de Ecuaciones diferenciales y compruebe

Encontrar la inversa de la T.L. de cada una de las funciones siguientes:

1.- $\frac{1}{s(s+1)}$ .... $\mathcal{L}^{-1}(\frac{1}{s(s+1)}) = 1 - e^{-t}$

$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$\mathcal{L}^{-1}(\frac{1}{s(s+1)}) = \mathcal{L}^{-1}(\frac{1}{s} - \frac{1}{s+1}) = \mathcal{L}^{-1}(\frac{1}{s}) - \mathcal{L}^{-1}(\frac{1}{s+1}) = 1 - e^{-t}$

2.- $\frac{3}{(s-1)^2}$ .... $\mathcal{L}^{-1}(\frac{3}{(s-1)^2}) = 3te^{t}$

$\mathcal{L}^{-1}(\frac{3}{(s-1)^2}) = 3\mathcal{L}^{-1}(\frac{1}{(s-1)^2}) = 3te^{t}$

3.- $\frac{1}{s(s+2)^2}$ .... $\mathcal{L}^{-1}(\frac{1}{s(s+2)^2}) = \frac{1}{4} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}$

$\frac{1}{s(s+2)^2} = \frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}$

$\mathcal{L}^{-1}(\frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}) = \frac{1}{4}\mathcal{L}^{-1}(\frac{1}{s}) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - \frac{1}{2}\mathcal{L}^{-1}(\frac{1}{(s+2)^2}$

....

4.- $\frac{5}{s^2(s-5)^2}$ .... $\mathcal{L}^{-1}(\frac{5}{s^2(s-5)^2}) = \frac{1}{5}t - \frac{2}{25}e^{5t} + \frac{1}{5}te^{5t} + \frac{2}{25}$

$\frac{5}{s^2(s-5)^2} = \frac{2}{25s} + \frac{1}{5s^2} - \frac{2}{25(s-5)} + \frac{1}{5(s-5)^2}$

$\mathcal{L}^{-1}(\frac{2}{25s} + \frac{1}{5s^2} - \frac{2}{25(s-5)} + \frac{1}{5(s-5)^2})$

$\frac{2}{25}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \frac{2}{25}\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) + \frac{1}{5}\mathcal{L}^{-1}(\frac{1}{(s-5)^2})$

Notabene:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$

5.- $\frac{1}{s^2+4s+29}$ .... $\mathcal{L}^{-1}(\frac{1}{s^2+4s+29}) = \frac{1}{5}(\sin 5t)e^{-2t}$

$\frac{1}{s^2+4s+4+25} = \frac{1}{(s+2)^2+5^2} = \frac{1}{5}\frac{5}{(s+2)^2+5^2}$

$$\frac{1}{5}\mathcal{L}^{-1}\left(\frac{5}{(s+2)^2+5^2}\right) = \frac{1}{5}(\sin 5t)e^{-2t}$$

notabene:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$

....

6.- $\frac{1}{s^2+3s}$ .... $\mathcal{L}^{-1}(\frac{1}{s^2+3s}) = \frac{1}{3} - \frac{1}{3}e^{-3t}$

$$\frac{1}{s^2+3s} = \frac{1}{3s} - \frac{1}{3(s+3)}$$

$$\mathcal{L}^{-1}(\frac{1}{s^2+3s}) = \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s}) - \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s+3}) = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

notabene:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
e^{at}	$\frac{1}{s-a}$	1	$\frac{1}{s}$

7.- $\frac{s}{s^2+1}$ .... $\mathcal{L}^{-1}(\frac{s}{s^2+1}) = \cos t$

....

8.- $\frac{s-1}{s^2+1}$ .... $\mathcal{L}^{-1}(\frac{s-1}{s^2+1}) = \cos t - \sin t$

$\frac{s-1}{s^2+1} = \frac{s}{s^2+1} - \frac{1}{s^2+1}$ ....$\mathcal{L}^{-1}(\frac{s-1}{s^2+1}) = \mathcal{L}^{-1}(\frac{s}{s^2+1}) - \mathcal{L}^{-1}(\frac{1}{s^2+1})$ en donde:

$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$ ....$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

9.- $\frac{s}{s^3+1}$ ....$\mathcal{L}^{-1}(\frac{s}{s^3+1}) = \frac{1}{3}e^{\frac{1}{2}t}\left(\cos\frac{1}{2}t\sqrt{3} + \sqrt{3}\sin\frac{1}{2}t\sqrt{3}\right) - \frac{1}{3}e^{-t}$

$$\frac{s}{s^3+1} = \frac{1}{3}\frac{s+1}{s^2-s+1} - \frac{1}{3(s+1)} = \frac{1}{3}\frac{s+1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{3(s+1)}$$

$$\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{3}\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{3(s+1)}$$

$$\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\sqrt{3}}{3}\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{3(s+1)}$$

$$\mathcal{L}^{-1}\left(\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{\sqrt{3}}{3}\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{3(s+1)}\right)$$

$$\frac{1}{3}\mathcal{L}^{-1}\left(\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right)+\frac{\sqrt{3}}{3}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right)-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\frac{1}{3}e^{\frac{1}{2}t}\cos\frac{1}{2}t\sqrt{3}+\frac{1}{3}\sqrt{3}\,e^{\frac{1}{2}t}\sin\frac{1}{2}t\sqrt{3}-\frac{1}{3}e^{-t}$$

$$\frac{1}{3}e^{\frac{1}{2}t}\left(\cos\frac{1}{2}t\sqrt{3}+\sqrt{3}\sin\frac{1}{2}t\sqrt{3}\right)-\frac{1}{3}e^{-t}$$

$$s^2-s+1=s^2-s+\left(\frac{1}{2}\right)^2+\frac{3}{4}$$

....

10.- $\frac{5}{s^2(s-5)}$ .... $\mathcal{L}^{-1}\left(\frac{5}{s^2(s-5)}\right)=\frac{1}{5}e^{5t}-t-\frac{1}{5}$

$$\frac{5}{s^2(s-5)}=\frac{1}{5(s-5)}-\frac{1}{s^2}-\frac{1}{5s}$$

$$\mathcal{L}^{-1}\left(\frac{5}{s^2(s-5)}\right)=\mathcal{L}^{-1}\left(\frac{1}{5(s-5)}-\frac{1}{s^2}-\frac{1}{5s}\right)$$

$$\frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s-5}\right)-\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)-\frac{1}{5}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$\mathcal{L}^{-1}\left(\frac{5}{s^2(s-5)}\right)=\frac{1}{5}e^{5t}-t-\frac{1}{5}$$

....

1.- $\mathcal{L}\left[e^{2t}(3\sin 4t-4\cos 4t)\right]=\mathcal{L}\left(e^{2t}3\sin 4t\right)-\mathcal{L}\left(e^{2t}4\cos 4t\right)$

$$\frac{12}{(s-2)^2+16}-4\frac{s-2}{(s-2)^2+16}=-4\frac{s-5}{s^2-4s+20}$$

Otra forma de ver la situación:

$\mathcal{L}(\sin 4t) = \dfrac{4}{s^2+16}$ ...... $\mathcal{L}(\cos 4t) = \dfrac{s}{s^2+16}$

ocupando el teorema: $\mathcal{L}(f(t)) = F(s) \rightarrow \mathcal{L}(e^{at}f(t)) = F(s-a)$

se tendrá:

$$\mathcal{L}\left(e^{2t}\sin 4t\right) = \frac{4}{(s-2)^2+16}$$

$$\mathcal{L}\left(e^{2t}\cos 4t\right) = \frac{s-2}{(s-2)^2+16}$$

finalmente: $\mathcal{L}\left[e^{2t}(3\sin 4t - 4\cos 4t)\right] = 3\mathcal{L}\left(e^{2t}\sin 4t\right) - 4\mathcal{L}\left(e^{2t}\cos 4t\right)$

$$3\frac{4}{(s-2)^2+16} - 4\frac{s-2}{(s-2)^2+16} = \frac{12}{(s-2)^2+16} - \frac{4s-8}{(s-2)^2+16}$$

$$\frac{12-(4s-8)}{(s-2)^2+16} = \frac{20-4s}{s^2-4s+20}$$

Se podría haber utilizado también el teorema siguiente:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$

que justamente se puede demostrar a partir del teorema:

$$\mathcal{L}(f(t)) = F(s) \rightarrow \mathcal{L}(e^{at}f(t)) = F(s-a)$$

y de: $\mathcal{L}(\sin kt) = \dfrac{k}{k^2+s^2}$ ...... $\mathcal{L}(\cos kt) = \dfrac{s}{k^2+s^2}$

2.- $\mathcal{L}\left[(t+2)^2e^t\right] = \mathcal{L}\left(4te^t + t^2e^t + 4e^t\right) = 4\mathcal{L}(te^t) + \mathcal{L}\left(t^2e^t\right) + 4\mathcal{L}(e^t)$

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer

el empleo de estas fórmulas nos saca del apuro, sin embargo,usaremos otra técnica,y es la que describe el siguiente teorema:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$

$$\mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\mathcal{L}(te^t) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

$$\mathcal{L}\left(t^2 e^t\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \frac{2}{(s-1)^3}$$

finalmente: $4\mathcal{L}(te^t) + \mathcal{L}\left(t^2 e^t\right) + 4\mathcal{L}(e^t)$

$$\frac{4}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{4}{s-1} = \frac{4s-4+2+4(s-1)^2}{(s-1)^3} = \frac{4s^2-4s+2}{(s-1)^3}$$

....

3.- Demuestre que $\mathcal{L}(t \sin t) = 2\frac{s}{\left(s^2+1\right)^2}$

a partir de $\mathcal{L}(\sin t) = \frac{1}{s^2+1} \rightarrow$

$$\mathcal{L}(t \sin t) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = 2\frac{s}{(s^2+1)^2}$$

....

4.- Demuestre que $\mathcal{L}\left(t^2 \sin t\right) = \frac{6s^2-2}{\left(s^2+1\right)^3}$

a partir de $\mathcal{L}(\sin t) = \frac{1}{s^2+1} \rightarrow$

$$\mathcal{L}\left(t^2 \sin t\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{6s^2-2}{\left(s^2+1\right)^3}$$

(la derivación se deja como ejercicio para el lector)

....

5.- Hallar $\mathcal{L}[t(3\sin 2t - 2\cos 2t)]$

recordar que $\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$ .....$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}$

$$\mathcal{L}[t(3\sin 2t - 2\cos 2t)] = 3\mathcal{L}(t\sin 2t) - 2\mathcal{L}(t\cos 2t)$$

$$3(-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right) - 2(-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$-3\left(\frac{-4s}{\left(s^2+4\right)^2}\right) + 2\left(\frac{s^2+4-s(2s)}{\left(s^2+4\right)^2}\right)$$

$$\frac{12s}{\left(s^2+4\right)^2} + 2\frac{4-s^2}{\left(s^2+4\right)^2} = \frac{12s+8-2s^2}{\left(s^2+4\right)^2}$$

....

6.- Calcular : $\mathcal{L}(t\cosh 3t)$

recordar que : $\cosh 3t = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{3t}$

$$\mathcal{L}(t\cosh 3t) = \mathcal{L}\left(t\left(\frac{1}{2}e^{-3t} + \frac{1}{2}e^{3t}\right)\right) = \mathcal{L}\left(\frac{1}{2}te^{-3t} + \frac{1}{2}te^{3t}\right)$$

$$\frac{1}{2}\mathcal{L}\left(te^{-3t}\right) + \frac{1}{2}\mathcal{L}\left(te^{3t}\right) = \frac{1}{2(s-3)^2} + \frac{1}{2(s+3)^2} = \frac{(s+3)^2+(s-3)^2}{2(s-3)^2(s+3)^2}$$

$$\frac{2s^2+18}{2(s-3)^2(s+3)^2} = \frac{s^2+9}{(s-3)^2(s+3)^2}$$

aplicando: $\mathcal{L}(te^{at}) = \frac{1}{(s-a)^2}$

fórmula que se obtiene por:

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$ y la aplicación del teorema anterior:

$$\mathcal{L}(te^{at}) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$$

7.- $\mathcal{L}(t\sinh 2t) = 4\dfrac{s}{\left(s^2-4\right)^2}$

recordar que : $\sinh 2t = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}$

$\mathcal{L}(t\sinh 2t) = \frac{1}{2}\mathcal{L}\left(te^{2t}\right) - \frac{1}{2}\mathcal{L}\left(te^{-2t}\right)$ aprovechando el trabajo anterior:

$$\mathcal{L}(t\sinh 2t) = \frac{1}{2}\frac{1}{(s-2)^2} - \frac{1}{2}\frac{1}{(s+2)^2} = \frac{(s+2)^2-(s-2)^2}{2(s-2)^2(s+2)^2}$$

$$\frac{8s}{2(s-2)^2(s+2)^2} = \frac{4s}{(s-2)^2(s+2)^2}$$

8.- $\mathcal{L}^{-1}(\dfrac{6s-4}{s^2-4s+20})$

$$\frac{6s-4}{s^2-4s+20} = \frac{6s-4}{s^2-4s+4+16} = \frac{6s-12+8}{(s-2)^2+16} = \frac{6(s-2)+8}{(s-2)^2+16}$$

$$\frac{6(s-2)}{(s-2)^2+16} + \frac{8}{(s-2)^2+16}$$

$$6\mathcal{L}^{-1}\left(\frac{(s-2)}{(s-2)^2+16}\right) + 2\mathcal{L}^{-1}\left(\frac{4}{(s-2)^2+16}\right)$$

$6(\cos 4t)e^{2t} + 2(\sin 4t)e^{2t}$

Notabene: Se ha aplicado el siguiente conocimiento:

$\mathcal{L}[f(t)] = F(s) \rightarrow \mathcal{L}[e^{at}f(t)] = F(s-a) \rightarrow$

$\mathcal{L}^{-1}[F(s)] = f(t) \rightarrow \mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$

9.- $\mathcal{L}\left[t^2\cos t\right] = 8\dfrac{s^3}{\left(s^2+1\right)^3} - 6\dfrac{s}{\left(s^2+1\right)^2}$

como $\mathcal{L}[\cos t] = \dfrac{s}{s^2+1}$

$$\mathcal{L}\left[t^2\cos t\right] = (-1)^2\frac{d^2}{ds^2}(\frac{s}{s^2+1}) = 8\frac{s^3}{\left(s^2+1\right)^3} - 6\frac{s}{\left(s^2+1\right)^2}$$

$$\frac{8s^3-6s\left(s^2+1\right)}{\left(s^2+1\right)^3} = \frac{2s^3-6s}{\left(s^2+1\right)^3}$$

10.- $\mathcal{L}\left[t^3 \cos t\right] = \frac{6}{\left(s^2+1\right)^2} - 48\frac{s^2}{\left(s^2+1\right)^3} + 48\frac{s^4}{\left(s^2+1\right)^4}$

$$\mathcal{L}\left[t^3 \cos t\right] = (-1)^3 \frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right) = \frac{6}{\left(s^2+1\right)^2} - \frac{48s^2}{\left(s^2+1\right)^3} + \frac{48s^4}{\left(s^2+1\right)^4}$$

(se deja para el lector el trabajo de derivar y reducir términos semejantes)

$$\frac{6\left(s^2+1\right)^2}{\left(s^2+1\right)^4} - \frac{48s^2\left(s^2+1\right)}{\left(s^2+1\right)^3} + \frac{48s^4}{\left(s^2+1\right)^4} = \frac{6s^4-36s^2+6}{\left(s^2+1\right)^4}$$

11.- Demuestre que : $\int_0^\infty (e^{-3t} t \sin t)dt = \frac{3}{50}$

Como $\mathcal{L}[t \sin t] = \int_0^\infty (e^{-st} t \sin t)dt$ .....aquí s =3

$\mathcal{L}[t \sin t] = 2\frac{s}{\left(s^2+1\right)^2}$; para s=3 se tiene: $2\frac{3}{\left(3^2+1\right)^2} = \frac{3}{50}$

12.- Calcular : $\mathcal{L}^{-1}\left[\frac{s+1}{6s^2+7s+2}\right]$

$$\frac{s+1}{6s^2+7s+2} = \frac{1}{2s+1} - \frac{1}{3s+2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{2s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{3s+2}\right]$$

$$\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{1}{2}}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{2}{3}}\right] = \frac{1}{2}e^{-\frac{1}{2}t} - \frac{1}{3}e^{-\frac{2}{3}t}$$